

A Brief Introduction to Riemannian Geometry

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1 An overview

1.1 The tangent bundle of a differential manifold

Let M be a smooth n -manifold: then it has charts $\phi_\alpha: U_\alpha \rightarrow M$ where U_α is an open subset of \mathbb{R}^n . Suppose that $\phi_\alpha(x_\alpha) = \phi_\beta(x_\beta) = p \in M$. Then $D(\phi_\beta^{-1} \circ \phi_\alpha)_{x_\alpha}$ is an isomorphism from $\mathbb{R}^n \equiv T_{x_\alpha}\mathbb{R}^n$ to $\mathbb{R}^n \equiv T_{x_\beta}\mathbb{R}^n$. We can think of the tangent space T_pM as $T_{x_\alpha}\mathbb{R}^n$, up to such isomorphisms.

Then any tensor field on M can be represented locally as a tensor field on the U_α that fits together the right way. For example, a vector field X on M is the same as a vector field X_α on every U_α , such that $(\phi_\beta^{-1} \circ \phi_\alpha)_*X_\alpha = X_\beta$ wherever the left side is well-defined.

1.2 The metric tensor for a Riemannian metric

A *Riemannian metric* on M is a smoothly varying choice of inner product \langle, \rangle on every tangent space T_pM of M . If $\phi: U \rightarrow M$ is a chart, then we thereby obtain an inner product \langle, \rangle_x for every point $x \in U$. To say that the inner product varies smoothly on M is the same as saying that $\langle \frac{d}{dx^i}, \frac{d}{dx^j} \rangle$ is a smooth function on U , for any pair $\frac{d}{dx^i}, \frac{d}{dx^j}$ of coordinate vector fields on U . In fact we often (but not so often in Do Carmo) write these functions as g^{ij} , where we think of g as the *metric tensor*, and we will sometimes write a Riemannian manifold as (M, g) .

If M is a submanifold of \mathbb{R}^N , then T_xM is a subspace of $T_x\mathbb{R}^N$; we can define a Riemannian metric on M by letting the inner product on each T_xM be the standard Euclidean inner product on \mathbb{R}^N . This is one of the original motivating examples for a Riemannian metric.

If $\eta: I \rightarrow M$ is a smooth path in M , we can compute the length of η (with respect to the Riemannian metric) as $\int_I \|\eta'\|$ (where $\|v\| = \sqrt{\langle v, v \rangle}$). We can then define a metric (in the sense of a metric space) on M by letting the distance between x and y be the infimum of the lengths of smooth paths connecting the two points.

1.3 The Levi-Cevita connection for a Riemannian metric

If X is a vector field on M , and f is a smooth function on M , then we can take the pointwise directional derivative Xf of f . If X and Y are vector fields on R^n , we can let $D_X Y$ be the directional derivative of Y in the direction of X . We compute this derivative by thinking of Y as n functions (the coordinate functions Y^i of Y), and letting $(D_X Y)^i = X(Y^i)$.

For a general manifold M , we have no natural choices of coordinate functions, so we have no way of computing $D_X Y$. But it turns out that when M has a Riemannian metric, there is a unique bilinear operator $\Delta: \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}$ (Where \mathcal{V} is the space of smooth vector fields on M) such that

1. $\Delta_{fV} W = f \Delta_V W$,
2. $\Delta_V fW = (D_V f)W + f \Delta_V W$
3. $D_V \langle X, Y \rangle = \langle \Delta_V X, Y \rangle + \langle X, \Delta_V Y \rangle$, and
4. $\Delta_X Y - \Delta_Y X = [X, Y]$.

This Δ is called the Levi-Cevita connection for (M, g) (where, as always $g(X, Y) = \langle X, Y \rangle$).

In the case where M is a submanifold of R^N , $\Delta_X Y$ (at $p \in M$) is the projection of $D_X Y$ onto the tangent space $T_p M$.

We observe that if $\eta: I \rightarrow M$ is a smooth (nonsingular) path, and $X: I \rightarrow TM$ is a vector field along I , then we can define $\frac{DX}{dt} \equiv \Delta_{\eta'} X$ by extending η' arbitrarily to all of M . If we are given a vector X_0 at $\eta(0)$, there is a unique vector field X along η for which $\frac{DX}{dt} = 0$; this vector field X is called the *parallel transport* of $X(0)$ along η .

1.4 Geodesics and Gauss's Lemma

We consider the "geodesic equation" $\Delta_{v(t)} v(t) = 0$, for a given $v(0) \in TM$. There is a unique solution (at least for t small) to this equation: we can think of this defining a flow on TM , which we call the geodesic flow. We can think of a geodesic as a path with "no acceleration or turning": it is a path η for which $\frac{D\eta'}{dt} = 0$.

It follows from property 3 of the connection that $\langle \eta', \eta' \rangle$ is constant, so a geodesic moves with constant speed. Moreover, a geodesic is locally length-minimizing: if $\eta: I \rightarrow M$ is a geodesic, and $x \in I$, and $y \in I$ is close to x , then η restricted to $[x, y]$ is the unique smooth path from $\eta(x)$ to $\eta(y)$ (up to reparametrization) whose length is equal to the distance between the two points. These two properties (constant speed and locally length-minimizing) characterize the geodesics.

We can also attempt to define the exponential map $\exp_p: T_p M \rightarrow M$ by letting $\exp_p(v_0) = y$, where y is the base point for $v(1) \in TM$, and $v(0) = v_0$, and $\Delta_{v(t)} v(t) = 0$. The exponential map will provide more or less natural coordinates around any point $p \in M$. I say "attempt to define" because in

general the exponential map is only defined in a neighborhood of zero; we are only guaranteed short-term solutions to the geodesic equation—but see Subsection 1.7 below.

1.5 Curvature

If X , Y and Z are vector fields on R^n , then $XYZ - YXZ = [X, Y]Z$, or equivalently $D_X D_Y Z - D_Y D_X Z = D_{[X, Y]}Z$. For a general Riemannian manifold M, g , this equation may fail to be true. We define the *curvature* $R(X, Y)Z$ as $\Delta_X \Delta_Y Z - \Delta_Y \Delta_X Z - \Delta_{[X, Y]}Z$. The remarkable thing is that the curvature is a tensor: the value of $R(X, Y)Z$ at a point p depends only on the values of X , Y , and Z at p . (Unlike $(\Delta_X Y)(p)$, which depends on the value of Y in a neighborhood of p .) We call this the curvature tensor: it describes, to first order, the amount that Z will be rotated when we parallel transport it around a small parallelogram generated by X and Y . The curvature tensor is naturally a $(3, 1)$ tensor (it takes 3 vectors as input, and outputs 1) but we can define the $(4, 0)$ tensor $R(X, Y, Z, W)$ by $R(X, Y, Z, W) = \langle R(X, Y)Z, W \rangle$. Then $R(X, Y, Z, W)$ has the following symmetry properties:

1. $R(X, Y, Z, W) = -R(Y, X, Z, W)$
2. $R(X, Y, Z, W) = R(Z, W, X, Y)$
3. $R(X, Y, Z, W) + R(X, W, Y, Z) + R(X, Z, W, Y) = 0$.

The properties 1 and 2 have the following corollary: let $Q \subset T_p M$ be a two-dimensional subspace, and let $X, Y \in T_p M$ be an orthonormal basis for Q . Then $R(X, Y, X, Y)$ depends only on Q . We call this the sectional curvature of M at p in the direction Q ; it is possible to reconstruct the whole curvature tensor from the sectional curvatures. When M is two-dimensional, there is of course one such Q at each $p \in M$, so the curvature at M is determined by a single number at each point, which is called the Gaussian curvature.

1.6 Jacobi Fields and Curvature

Suppose that the smooth map $f: A \times B \rightarrow M$ (for A and B intervals in \mathbb{R}^n) is such that $f(\cdot, y)$ is a geodesic for each value of y (more precisely, it is the projection of a geodesic from TM to M). Then $J \equiv \frac{\partial f}{\partial x}(x, y_0)$ is a vector field along $f(\cdot, y_0)$ that describes an infinitesimal variation of geodesics: we say that J is a *Jacobi field*. It satisfies the Jacobi equation

$$\frac{D^2 J}{dt^2} + R(\gamma'(t), J(t))\gamma'(t) = 0 \quad (1)$$

(where $\gamma(t) \equiv f(t, y_0)$).

1.7 Hopf-Rinow and Hadamard

We prove that the following three statements (for M, g) are equivalent:

1. The exponential map is defined on all of TM .
2. There exists a point $p \in M$ such that the exponential map is defined on all of T_pM .
3. The metric induced by the Riemannian metric is complete.

These statements, in turn, imply that any two points of M can be connected by a geodesic. Some portion of this, or all of it, is known as the Hopf-Rinow Theorem.

Now suppose that M is complete and simply connected, and all of the sectional curvatures are nonpositive. We prove a theorem of Hadamard, which says that for any $p \in M$, the map $\exp_p: T_pM \rightarrow M$ is a diffeomorphism. Some hint of how this theorem is proven is found in equation 1, which implies that

$$\left\langle \frac{D^2 J}{dt^2}, J(t) \right\rangle + R(\gamma'(t), J(t), \gamma'(t), J(t)) = 0$$

which in turn implies, when sectional curvatures are nonpositive, that $\frac{D^2}{dt^2} \langle J(t), J(t) \rangle \geq 0$. Hence if $J(0) = 0$, and $\frac{D}{dt} \langle J(t), J(t) \rangle > 0$ when $t = 0$, then $\|J(t)\| > 0$ for all $t > 0$. This in turn implies that the exponential map is a local diffeomorphism.